

MATRICES

❖ *The essence of Mathematics lies in its freedom.* — CANTOR ❖

3.1 Introduction

The knowledge of matrices is necessary in various branches of mathematics. Matrices are one of the most powerful tools in mathematics. This mathematical tool simplifies our work to a great extent when compared with other straight forward methods. The evolution of concept of matrices is the result of an attempt to obtain compact and simple methods of solving system of linear equations. Matrices are not only used as a representation of the coefficients in system of linear equations, but utility of matrices far exceeds that use. Matrix notation and operations are used in electronic spreadsheet programs for personal computer, which in turn is used in different areas of business and science like budgeting, sales projection, cost estimation, analysing the results of an experiment etc. Also, many physical operations such as magnification, rotation and reflection through a plane can be represented mathematically by matrices. Matrices are also used in cryptography. This mathematical tool is not only used in certain branches of sciences, but also in genetics, economics, sociology, modern psychology and industrial management.

In this chapter, we shall find it interesting to become acquainted with the fundamentals of matrix and matrix algebra.

3.2 Matrix

Suppose we wish to express the information that Radha has 15 notebooks. We may express it as [15] with the understanding that the number inside [] is the number of notebooks that Radha has. Now, if we have to express that Radha has 15 notebooks and 6 pens. We may express it as [15 6] with the understanding that first number inside [] is the number of notebooks while the other one is the number of pens possessed by Radha. Let us now suppose that we wish to express the information of possession

of notebooks and pens by Radha and her two friends Fauzia and Simran which is as follows:

Radha	has	15	notebooks	and	6 pens,
Fauzia	has	10	notebooks	and	2 pens,
Simran	has	13	notebooks	and	5 pens.

Now this could be arranged in the tabular form as follows:

	Notebooks	Pens
Radha	15	6
Fauzia	10	2
Simran	13	5

and this can be expressed as

$$\begin{array}{c}
 \left[\begin{array}{c} 15 \\ 10 \\ 13 \end{array} \right. \quad \left. \begin{array}{c} 6 \\ 2 \\ 5 \end{array} \right] \\
 \uparrow \qquad \qquad \uparrow \\
 \text{First} \qquad \text{Second} \\
 \text{Column} \qquad \text{Column}
 \end{array}
 \begin{array}{l}
 \leftarrow \text{ First row} \\
 \leftarrow \text{ Second row} \\
 \leftarrow \text{ Third row}
 \end{array}$$

or

	Radha	Fauzia	Simran
Notebooks	15	10	13
Pens	6	2	5

which can be expressed as:

$$\begin{array}{c}
 \left[\begin{array}{ccc} 15 & 10 & 13 \\ 6 & 2 & 5 \end{array} \right] \\
 \uparrow \qquad \uparrow \qquad \uparrow \\
 \text{First} \quad \text{Second} \quad \text{Third} \\
 \text{Column} \quad \text{Column} \quad \text{Column}
 \end{array}
 \begin{array}{l}
 \leftarrow \text{ First row} \\
 \leftarrow \text{ Second row}
 \end{array}$$

In the first arrangement the entries in the first column represent the number of note books possessed by Radha, Fauzia and Simran, respectively and the entries in the second column represent the number of pens possessed by Radha, Fauzia and Simran,

respectively. Similarly, in the second arrangement, the entries in the first row represent the number of notebooks possessed by Radha, Fauzia and Simran, respectively. The entries in the second row represent the number of pens possessed by Radha, Fauzia and Simran, respectively. An arrangement or display of the above kind is called a *matrix*. Formally, we define matrix as:

Definition 1 A *matrix* is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements or the entries of the matrix.

We denote matrices by capital letters. The following are some examples of matrices:

$$A = \begin{bmatrix} -2 & 5 \\ 0 & \sqrt{5} \\ 3 & 6 \end{bmatrix}, B = \begin{bmatrix} 2+i & 3 & -\frac{1}{2} \\ 3.5 & -1 & 2 \\ \sqrt{3} & 5 & \frac{5}{7} \end{bmatrix}, C = \begin{bmatrix} 1+x & x^3 & 3 \\ \cos x & \sin x + 2 & \tan x \end{bmatrix}$$

In the above examples, the horizontal lines of elements are said to constitute, **rows** of the matrix and the vertical lines of elements are said to constitute, **columns** of the matrix. Thus A has 3 rows and 2 columns, B has 3 rows and 3 columns while C has 2 rows and 3 columns.

3.2.1 Order of a matrix

A matrix having m rows and n columns is called a matrix of *order* $m \times n$ or simply $m \times n$ matrix (read as an m by n matrix). So referring to the above examples of matrices, we have A as 3×2 matrix, B as 3×3 matrix and C as 2×3 matrix. We observe that A has $3 \times 2 = 6$ elements, B and C have 9 and 6 elements, respectively.


In general, an $m \times n$ matrix has the following rectangular array:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

or $A = [a_{ij}]_{m \times n}$, $1 \leq i \leq m$, $1 \leq j \leq n$ $i, j \in \mathbb{N}$

Thus the i^{th} row consists of the elements $a_{i1}, a_{i2}, a_{i3}, \dots, a_{in}$, while the j^{th} column consists of the elements $a_{1j}, a_{2j}, a_{3j}, \dots, a_{mj}$,

In general a_{ij} , is an element lying in the i^{th} row and j^{th} column. We can also call it as the $(i, j)^{\text{th}}$ element of A. The number of elements in an $m \times n$ matrix will be equal to mn .

 **Note** In this chapter

1. We shall follow the notation, namely $A = [a_{ij}]_{m \times n}$ to indicate that A is a matrix of order $m \times n$.
2. We shall consider only those matrices whose elements are real numbers or functions taking real values.

We can also represent any point (x, y) in a plane by a matrix (column or row) as $\begin{bmatrix} x \\ y \end{bmatrix}$ (or $[x, y]$). For example point P(0, 1) as a matrix representation may be given as

$$P = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ or } [0 \ 1].$$

Observe that in this way we can also express the vertices of a closed rectilinear figure in the form of a matrix. For example, consider a quadrilateral ABCD with vertices A (1, 0), B (3, 2), C (1, 3), D (-1, 2).

Now, quadrilateral ABCD in the matrix form, can be represented as

$$X = \begin{matrix} & \text{A} & \text{B} & \text{C} & \text{D} \\ \begin{bmatrix} 1 & 3 & 1 & -1 \\ 0 & 2 & 3 & 2 \end{bmatrix}_{2 \times 4} & & & & \end{matrix} \quad \text{or} \quad \begin{matrix} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{matrix} \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 1 & 3 \\ -1 & 2 \end{bmatrix}_{4 \times 2}$$

Thus, matrices can be used as representation of vertices of geometrical figures in a plane.

Now, let us consider some examples.

Example 1 Consider the following information regarding the number of men and women workers in three factories I, II and III

	Men workers	Women workers
I	30	25
II	25	31
III	27	26

Represent the above information in the form of a 3×2 matrix. What does the entry in the third row and second column represent?

Solution The information is represented in the form of a 3×2 matrix as follows:

$$A = \begin{bmatrix} 30 & 25 \\ 25 & 31 \\ 27 & 26 \end{bmatrix}$$

The entry in the third row and second column represents the number of women workers in factory III.

Example 2 If a matrix has 8 elements, what are the possible orders it can have?

Solution We know that if a matrix is of order $m \times n$, it has mn elements. Thus, to find all possible orders of a matrix with 8 elements, we will find all ordered pairs of natural numbers, whose product is 8.

Thus, all possible ordered pairs are (1, 8), (8, 1), (4, 2), (2, 4)

Hence, possible orders are 1×8 , 8×1 , 4×2 , 2×4

Example 3 Construct a 3×2 matrix whose elements are given by $a_{ij} = \frac{1}{2}|i - 3j|$.

Solution In general a 3×2 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$.

Now $a_{ij} = \frac{1}{2}|i - 3j|$, $i = 1, 2, 3$ and $j = 1, 2$.

Therefore $a_{11} = \frac{1}{2}|1 - 3 \times 1| = 1$ $a_{12} = \frac{1}{2}|1 - 3 \times 2| = \frac{5}{2}$

$a_{21} = \frac{1}{2}|2 - 3 \times 1| = \frac{1}{2}$ $a_{22} = \frac{1}{2}|2 - 3 \times 2| = 2$

$a_{31} = \frac{1}{2}|3 - 3 \times 1| = 0$ $a_{32} = \frac{1}{2}|3 - 3 \times 2| = \frac{3}{2}$

Hence the required matrix is given by $A = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \\ 0 & \frac{3}{2} \end{bmatrix}$.

3.3 Types of Matrices

In this section, we shall discuss different types of matrices.

(i) **Column matrix**

A matrix is said to be a *column matrix* if it has only one column.

For example, $A = \begin{bmatrix} 0 \\ \sqrt{3} \\ -1 \\ 1/2 \end{bmatrix}$ is a column matrix of order 4×1 .

In general, $A = [a_{ij}]_{m \times 1}$ is a column matrix of order $m \times 1$.

(ii) **Row matrix**

A matrix is said to be a *row matrix* if it has only one row.

For example, $B = \left[-\frac{1}{2} \quad \sqrt{5} \quad 2 \quad 3 \right]_{1 \times 4}$ is a row matrix.

In general, $B = [b_{ij}]_{1 \times n}$ is a row matrix of order $1 \times n$.

(iii) **Square matrix**

A matrix in which the number of rows are equal to the number of columns, is said to be a *square matrix*. Thus an $m \times n$ matrix is said to be a square matrix if $m = n$ and is known as a square matrix of order ' n '.

For example $A = \begin{bmatrix} 3 & -1 & 0 \\ \frac{3}{2} & 3\sqrt{2} & 1 \\ 4 & 3 & -1 \end{bmatrix}$ is a square matrix of order 3.

In general, $A = [a_{ij}]_{m \times m}$ is a square matrix of order m .

 **Note** If $A = [a_{ij}]$ is a square matrix of order n , then elements (entries) $a_{11}, a_{22}, \dots, a_{nn}$

are said to constitute the *diagonal*, of the matrix A . Thus, if $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 4 & -1 \\ 3 & 5 & 6 \end{bmatrix}$.

Then the elements of the diagonal of A are 1, 4, 6.

(iv) Diagonal matrix

A square matrix $B = [b_{ij}]_{m \times m}$ is said to be a *diagonal matrix* if all its non diagonal elements are zero, that is a matrix $B = [b_{ij}]_{m \times m}$ is said to be a diagonal matrix if $b_{ij} = 0$, when $i \neq j$.

For example, $A = [4]$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, $C = \begin{bmatrix} -1.1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, are diagonal matrices

of order 1, 2, 3, respectively.

(v) Scalar matrix

A diagonal matrix is said to be a *scalar matrix* if its diagonal elements are equal, that is, a square matrix $B = [b_{ij}]_{n \times n}$ is said to be a scalar matrix if

$$\begin{aligned} b_{ij} &= 0, & \text{when } i &\neq j \\ b_{ij} &= k, & \text{when } i &= j, \text{ for some constant } k. \end{aligned}$$

For example

$$A = [3], \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

are scalar matrices of order 1, 2 and 3, respectively.

(vi) Identity matrix

A square matrix in which elements in the diagonal are all 1 and rest are all zero is called an *identity matrix*. In other words, the square matrix $A = [a_{ij}]_{n \times n}$ is an

identity matrix, if $a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

We denote the identity matrix of order n by I_n . When order is clear from the context, we simply write it as I .

For example $[1]$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 1, 2 and 3,

respectively.

Observe that a scalar matrix is an identity matrix when $k = 1$. But every identity matrix is clearly a scalar matrix.

(vii) **Zero matrix**

A matrix is said to be *zero matrix* or *null matrix* if all its elements are zero.

For example, $[0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $[0, 0]$ are all zero matrices. We denote zero matrix by O . Its order will be clear from the context.

3.3.1 Equality of matrices

Definition 2 Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if

- (i) they are of the same order
- (ii) each element of A is equal to the corresponding element of B , that is $a_{ij} = b_{ij}$ for all i and j .

For example, $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are equal matrices but $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are not equal matrices. Symbolically, if two matrices A and B are equal, we write $A = B$.

If $\begin{bmatrix} x & y \\ z & a \\ b & c \end{bmatrix} = \begin{bmatrix} -1.5 & 0 \\ 2 & \sqrt{6} \\ 3 & 2 \end{bmatrix}$, then $x = -1.5, y = 0, z = 2, a = \sqrt{6}, b = 3, c = 2$

Example 4 If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$

Find the values of a, b, c, x, y and z .

Solution As the given matrices are equal, therefore, their corresponding elements must be equal. Comparing the corresponding elements, we get

$$\begin{aligned} x + 3 &= 0, & z + 4 &= 6, & 2y - 7 &= 3y - 2 \\ a - 1 &= -3, & 0 &= 2c + 2 & b - 3 &= 2b + 4, \end{aligned}$$

Simplifying, we get

$$a = -2, b = -7, c = -1, x = -3, y = -5, z = 2$$

Example 5 Find the values of a, b, c , and d from the following equation:

$$\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$$

Solution By equality of two matrices, equating the corresponding elements, we get

$$\begin{aligned} 2a + b &= 4 & 5c - d &= 11 \\ a - 2b &= -3 & 4c + 3d &= 24 \end{aligned}$$

Solving these equations, we get

$$a = 1, b = 2, c = 3 \text{ and } d = 4$$

EXERCISE 3.1

1. In the matrix $A = \begin{bmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & \frac{5}{2} & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{bmatrix}$, write:

- (i) The order of the matrix, (ii) The number of elements,
 (iii) Write the elements a_{13} , a_{21} , a_{33} , a_{24} , a_{23} .
2. If a matrix has 24 elements, what are the possible orders it can have? What, if it has 13 elements?
3. If a matrix has 18 elements, what are the possible orders it can have? What, if it has 5 elements?
4. Construct a 2×2 matrix, $A = [a_{ij}]$, whose elements are given by:

$$(i) a_{ij} = \frac{(i+j)^2}{2} \quad (ii) a_{ij} = \frac{i}{j} \quad (iii) a_{ij} = \frac{(i+2j)^2}{2}$$

5. Construct a 3×4 matrix, whose elements are given by:

$$(i) a_{ij} = \frac{1}{2} |-3i + j| \quad (ii) a_{ij} = 2i - j$$

6. Find the values of x , y and z from the following equations:

$$(i) \begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix} \quad (ii) \begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix} \quad (iii) \begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$$

7. Find the value of a , b , c and d from the equation:

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

8. $A = [a_{ij}]_{m \times n}$ is a square matrix, if
 (A) $m < n$ (B) $m > n$ (C) $m = n$ (D) None of these
9. Which of the given values of x and y make the following pair of matrices equal
 $\begin{bmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{bmatrix}, \begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix}$
 (A) $x = \frac{-1}{3}, y = 7$ (B) Not possible to find
 (C) $y = 7, x = \frac{-2}{3}$ (D) $x = \frac{1}{3}, y = \frac{2}{3}$
10. The number of all possible matrices of order 3×3 with each entry 0 or 1 is:
 (A) 27 (B) 18 (C) 81 (D) 512

3.4 Operations on Matrices

In this section, we shall introduce certain operations on matrices, namely, addition of matrices, multiplication of a matrix by a scalar, difference and multiplication of matrices.

3.4.1 Addition of matrices

Suppose Fatima has two factories at places A and B. Each factory produces sport shoes for boys and girls in three different price categories labelled 1, 2 and 3. The quantities produced by each factory are represented as matrices given below:

	Factory at A		Factory at B	
	Boys	Girls	Boys	Girls
1	80	60	90	50
2	75	65	70	55
3	90	85	75	75

Suppose Fatima wants to know the total production of sport shoes in each price category. Then the total production

In category 1 : for boys $(80 + 90)$, for girls $(60 + 50)$

In category 2 : for boys $(75 + 70)$, for girls $(65 + 55)$

In category 3 : for boys $(90 + 75)$, for girls $(85 + 75)$

This can be represented in the matrix form as $\begin{bmatrix} 80+90 & 60+50 \\ 75+70 & 65+55 \\ 90+75 & 85+75 \end{bmatrix}$.

This new matrix is the **sum** of the above two matrices. We observe that the sum of two matrices is a matrix obtained by adding the corresponding elements of the given matrices. Furthermore, the two matrices have to be of the same order.

Thus, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ is a 2×3 matrix and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$ is another

2×3 matrix. Then, we define $A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix}$.

In general, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of the same order, say $m \times n$. Then, the sum of the two matrices A and B is *defined* as a matrix $C = [c_{ij}]_{m \times n}$, where $c_{ij} = a_{ij} + b_{ij}$, for all possible values of i and j .

Example 6 Given $A = \begin{bmatrix} \sqrt{3} & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & \sqrt{5} & 1 \\ -2 & 3 & \frac{1}{2} \end{bmatrix}$, find $A + B$

Since A, B are of the same order 2×3 . Therefore, addition of A and B is defined and is given by

$$A + B = \begin{bmatrix} 2 + \sqrt{3} & 1 + \sqrt{5} & 1 - 1 \\ 2 - 2 & 3 + 3 & 0 + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 + \sqrt{3} & 1 + \sqrt{5} & 0 \\ 0 & 6 & \frac{1}{2} \end{bmatrix}$$

Note

1. We emphasise that if A and B are not of the same order, then $A + B$ is not defined. For example if $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$, then $A + B$ is not defined.
2. We may observe that addition of matrices is an example of binary operation on the set of matrices of the same order.

3.4.2 Multiplication of a matrix by a scalar

Now suppose that Fatima has doubled the production at a factory A in all categories (refer to 3.4.1).

Previously quantities (in standard units) produced by factory A were

$$\begin{array}{cc} & \text{Boys} & \text{Girls} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cc} 80 & 60 \\ 75 & 65 \\ 90 & 85 \end{array} \right] \end{array}$$

Revised quantities produced by factory A are as given below:

$$\begin{array}{cc} & \text{Boys} & \text{Girls} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cc} 2 \times 80 & 2 \times 60 \\ 2 \times 75 & 2 \times 65 \\ 2 \times 90 & 2 \times 85 \end{array} \right] \end{array}$$

This can be represented in the matrix form as $\begin{bmatrix} 160 & 120 \\ 150 & 130 \\ 180 & 170 \end{bmatrix}$. We observe that

the new matrix is obtained by multiplying each element of the previous matrix by 2.

In general, we may define *multiplication of a matrix* by a scalar as follows: if $A = [a_{ij}]_{m \times n}$ is a matrix and k is a scalar, then kA is another matrix which is obtained by multiplying each element of A by the scalar k .

In other words, $kA = k[a_{ij}]_{m \times n} = [k(a_{ij})]_{m \times n}$, that is, $(i, j)^{\text{th}}$ element of kA is ka_{ij} for all possible values of i and j .

For example, if $A = \begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix}$, then

$$3A = 3 \begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 4.5 \\ 3\sqrt{5} & 21 & -9 \\ 6 & 0 & 15 \end{bmatrix}$$

Negative of a matrix The negative of a matrix is denoted by $-A$. We define $-A = (-1) A$.

For example, let

$$A = \begin{bmatrix} 3 & 1 \\ -5 & x \end{bmatrix}, \text{ then } -A \text{ is given by}$$

$$-A = (-1)A = (-1) \begin{bmatrix} 3 & 1 \\ -5 & x \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 5 & -x \end{bmatrix}$$

Difference of matrices If $A = [a_{ij}]$, $B = [b_{ij}]$ are two matrices of the same order, say $m \times n$, then difference $A - B$ is defined as a matrix $D = [d_{ij}]$, where $d_{ij} = a_{ij} - b_{ij}$, for all value of i and j . In other words, $D = A - B = A + (-1)B$, that is sum of the matrix A and the matrix $-B$.

Example 7 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$, then find $2A - B$.

Solution We have

$$\begin{aligned} 2A - B &= 2 \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2-3 & 4+1 & 6-3 \\ 4+1 & 6+0 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 5 & 3 \\ 5 & 6 & 0 \end{bmatrix} \end{aligned}$$

3.4.3 Properties of matrix addition

The addition of matrices satisfy the following properties:

- (i) **Commutative Law** If $A = [a_{ij}]$, $B = [b_{ij}]$ are matrices of the same order, say $m \times n$, then $A + B = B + A$.

$$\begin{aligned} \text{Now } A + B &= [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] \\ &= [b_{ij} + a_{ij}] \text{ (addition of numbers is commutative)} \\ &= ([b_{ij}] + [a_{ij}]) = B + A \end{aligned}$$

- (ii) **Associative Law** For any three matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$ of the same order, say $m \times n$, $(A + B) + C = A + (B + C)$.

$$\begin{aligned} \text{Now } (A + B) + C &= ([a_{ij}] + [b_{ij}]) + [c_{ij}] \\ &= [a_{ij} + b_{ij}] + [c_{ij}] = [(a_{ij} + b_{ij}) + c_{ij}] \\ &= [a_{ij} + (b_{ij} + c_{ij})] \quad \text{(Why?)} \\ &= [a_{ij}] + [(b_{ij} + c_{ij})] = [a_{ij}] + ([b_{ij}] + [c_{ij}]) = A + (B + C) \end{aligned}$$

- (iii) **Existence of additive identity** Let $A = [a_{ij}]$ be an $m \times n$ matrix and O be an $m \times n$ zero matrix, then $A + O = O + A = A$. In other words, O is the additive identity for matrix addition.
- (iv) **The existence of additive inverse** Let $A = [a_{ij}]_{m \times n}$ be any matrix, then we have another matrix as $-A = [-a_{ij}]_{m \times n}$ such that $A + (-A) = (-A) + A = O$. So $-A$ is the additive inverse of A or negative of A .

3.4.4 Properties of scalar multiplication of a matrix

If $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of the same order, say $m \times n$, and k and l are scalars, then

- (i) $k(A + B) = kA + kB$, (ii) $(k + l)A = kA + lA$
- (ii) $k(A + B) = k([a_{ij}] + [b_{ij}])$
 $= k[a_{ij} + b_{ij}] = [k(a_{ij} + b_{ij})] = [(ka_{ij}) + (kb_{ij})]$
 $= [ka_{ij}] + [kb_{ij}] = k[a_{ij}] + k[b_{ij}] = kA + kB$
- (iii) $(k + l)A = (k + l)[a_{ij}]$
 $= [(k + l)a_{ij}] + [ka_{ij}] + [la_{ij}] = k[a_{ij}] + l[a_{ij}] = kA + lA$

Example 8 If $A = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$, then find the matrix X , such that

$2A + 3X = 5B$.

Solution We have $2A + 3X = 5B$

- or $2A + 3X - 2A = 5B - 2A$
- or $2A - 2A + 3X = 5B - 2A$ (Matrix addition is commutative)
- or $O + 3X = 5B - 2A$ ($-2A$ is the additive inverse of $2A$)
- or $3X = 5B - 2A$ (O is the additive identity)
- or $X = \frac{1}{3}(5B - 2A)$

or $X = \frac{1}{3} \left(5 \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix} - 2 \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix} \right) = \frac{1}{3} \left(\begin{bmatrix} 10 & -10 \\ 20 & 10 \\ -25 & 5 \end{bmatrix} + \begin{bmatrix} -16 & 0 \\ -8 & 4 \\ -6 & -12 \end{bmatrix} \right)$

$$= \frac{1}{3} \begin{bmatrix} 10-16 & -10+0 \\ 20-8 & 10+4 \\ -25-6 & 5-12 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -6 & -10 \\ 12 & 14 \\ -31 & -7 \end{bmatrix} = \begin{bmatrix} -2 & \frac{-10}{3} \\ 4 & \frac{14}{3} \\ \frac{-31}{3} & \frac{-7}{3} \end{bmatrix}$$

Example 9 Find X and Y , if $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

Solution We have $(X + Y) + (X - Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

or $(X + X) + (Y - Y) = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} \Rightarrow 2X = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}$

or $X = \frac{1}{2} \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$

Also $(X + Y) - (X - Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

or $(X - X) + (Y + Y) = \begin{bmatrix} 5-3 & 2-6 \\ 0 & 9+1 \end{bmatrix} \Rightarrow 2Y = \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix}$

or $Y = \frac{1}{2} \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$

Example 10 Find the values of x and y from the following equation:

$$2 \begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Solution We have

$$2 \begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} 2x+3 & 10-4 \\ 14+1 & 2y-6+2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\text{or } \qquad 2x + 3 = 7 \qquad \text{and} \qquad 2y - 4 = 14 \qquad (\text{Why?})$$

$$\text{or } \qquad 2x = 7 - 3 \qquad \text{and} \qquad 2y = 18$$

$$\text{or } \qquad x = \frac{4}{2} \qquad \text{and} \qquad y = \frac{18}{2}$$

$$\text{i.e.} \qquad x = 2 \qquad \text{and} \qquad y = 9.$$

Example 11 Two farmers Ramkishan and Gurcharan Singh cultivates only three varieties of rice namely Basmati, Permal and Naura. The sale (in Rupees) of these varieties of rice by both the farmers in the month of September and October are given by the following matrices A and B.

September Sales (in Rupees)

$$A = \begin{bmatrix} \text{Basmati} & \text{Permal} & \text{Naura} \\ 10,000 & 20,000 & 30,000 \\ 50,000 & 30,000 & 10,000 \end{bmatrix} \begin{matrix} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{matrix}$$

October Sales (in Rupees)

$$B = \begin{bmatrix} \text{Basmati} & \text{Permal} & \text{Naura} \\ 5000 & 10,000 & 6000 \\ 20,000 & 10,000 & 10,000 \end{bmatrix} \begin{matrix} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{matrix}$$

- (i) Find the combined sales in September and October for each farmer in each variety.
- (ii) Find the decrease in sales from September to October.
- (iii) If both farmers receive 2% profit on gross sales, compute the profit for each farmer and for each variety sold in October.

Solution

- (i) Combined sales in September and October for each farmer in each variety is given by

$$A + B = \begin{bmatrix} \text{Basmati} & \text{Permal} & \text{Naura} \\ 15,000 & 30,000 & 36,000 \\ 70,000 & 40,000 & 20,000 \end{bmatrix} \begin{matrix} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{matrix}$$

