Chapter 2

INVERSE TRIGONOMETRIC FUNCTIONS

2.1 Overview

2.1.1 *Inverse function*

Inverse of a function 'f' exists, if the function is one-one and onto, i.e, bijective. Since trigonometric functions are many-one over their domains, we restrict their domains and co-domains in order to make them one-one and onto and then find their inverse. The domains and ranges (principal value branches) of inverse trigonometric functions are given below:

Functions	**Domain**	**Range (Principal value branches)**
$y = \sin^{-1}x$	$[-1,1]$	$\frac{-\pi}{2},\frac{\pi}{2}$
$y = \cos^{-1}x$	$[-1,1]$	$[0,\pi]$
$y = \text{cosec}^{-1}x$	$\mathbf{R}-(-1,1)$	$\frac{-\pi}{2},\frac{\pi}{2} - \{0\}$
$y = \sec^{-1}x$	$\mathbf{R}-(-1,1)$	$[0,\pi] - \frac{\pi}{2}$
$y = \tan^{-1}x$	$\mathbf{R}$	$\frac{-\pi}{2},\frac{\pi}{2}$
$y = \cot^{-1}x$	$\mathbf{R}$	$(0,\pi)$

Notes:

(i) The symbol $\sin^{-1}x$ should not be confused with $(\sin x)^{-1}$. Infact $\sin^{-1}x$ is an angle, the value of whose sine is x, similarly for other trigonometric functions.

(ii) The smallest numerical value, either positive or negative, of θ is called the principal value of the function.

(iii) Whenever no branch of an inverse trigonometric function is mentioned, we mean the principal value branch. The value of the inverse trigonometic function which lies in the range of principal branch is its principal value.

2.1.2 *Graph of an inverse trigonometric function*

The graph of an inverse trigonometric function can be obtained from the graph of original function by interchanging x-axis and y-axis, i.e, if (a, b) is a point on the graph of trigonometric function, then (b, a) becomes the corresponding point on the graph of its inverse trigonometric function.

It can be shown that the graph of an inverse function can be obtained from the corresponding graph of original function as a mirror image (i.e., reflection) along the line $y = x$.

2.1.3 *Properties of inverse trigonometric functions*

1. $\sin^{-1}(\sin x) = x$: $x \quad \frac{-}{2}, \frac{}{2}$

$\cos^{-1}(\cos x) = x$: $x \quad [0, \ ]$

$\tan^{-1}(\tan x) = x$: $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$\cot^{-1}(\cot x) = x$: $x \in (0, \pi)$

$\sec^{-1}(\sec x) = x$: $x \quad [0, \pi] - \frac{\pi}{2}$

$\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x$: $x \quad \frac{-\pi}{2}, \frac{\pi}{2} \quad - \{0\}$

2. $\sin(\sin^{-1} x) = x$: $x \in [-1, 1]$

$\cos(\cos^{-1} x) = x$: $x \in [-1, 1]$

$\tan(\tan^{-1} x) = x$: $x \in \mathbf{R}$

$\cot(\cot^{-1} x) = x$: $x \in \mathbf{R}$

$\sec(\sec^{-1} x) = x$: $x \in \mathbf{R} - (-1, 1)$

$\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$: $x \in \mathbf{R} - (-1, 1)$

3. $\sin^{-1}\frac{1}{x} \quad \operatorname{cosec}^{-1} x$: $x \in \mathbf{R} - (-1, 1)$

$\cos^{-1}\frac{1}{x} \quad \sec^{-1} x$: $x \in \mathbf{R} - (-1, 1)$

$\tan^{-1}\dfrac{1}{x} \quad \cot^{-1}x$: $x > 0$

$= -\pi + \cot^{-1}x$: $x < 0$

4. $\sin^{-1}(-x) = -\sin^{-1}x$: $x \in [-1,1]$

$\cos^{-1}(-x) = \pi - \cos^{-1}x$: $x \in [-1,1]$

$\tan^{-1}(-x) = -\tan^{-1}x$: $x \in \mathbf{R}$

$\cot^{-1}(-x) = \pi - \cot^{-1}x$: $x \in \mathbf{R}$

$\sec^{-1}(-x) = \pi - \sec^{-1}x$: $x \in \mathbf{R} - (-1,1)$

$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$: $x \in \mathbf{R} - (-1,1)$

5. $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$: $x \in [-1,1]$

$\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}$: $x \in \mathbf{R}$

$\sec^{-1}x + \text{cosec}^{-1}x = \dfrac{\pi}{2}$: $x \in \mathbf{R} - [-1,1]$

6. $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x \quad y}{1-xy}$: $xy < 1$

$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right); xy > -1$

7. $2\tan^{-1}x = \sin^{-1}\dfrac{2x}{1 \quad x^2}$: $-1 \le x \le 1$

$2\tan^{-1}x = \cos^{-1}\dfrac{1-x^2}{1 \quad x^2}$: $x \ge 0$

$2\tan^{-1}x = \tan^{-1}\dfrac{2x}{1-x^2}$: $-1 < x < 1$

2.2 Solved Examples

Short Answer (S.A.)

Example 1 Find the principal value of $\cos^{-1}x$, for $x = \dfrac{\sqrt{3}}{2}$.

Solution If $\cos^{-1} \frac{\sqrt{3}}{2} = \theta$, then $\cos\theta = \frac{\sqrt{3}}{2}$.

Since we are considering principal branch, $\theta \in [0, \pi]$. Also, since $\frac{\sqrt{3}}{2} > 0$, θ being in the first quadrant, hence $\cos^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{6}$.

Example 2 Evaluate $\tan^{-1} \sin \frac{-\pi}{2}$.

Solution $\tan^{-1} \sin \frac{-\pi}{2} = \tan^{-1}\left(-\sin\left(\frac{\pi}{2}\right)\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$.

Example 3 Find the value of $\cos^{-1} \cos\frac{13\pi}{6}$.

Solution $\cos^{-1} \cos\frac{13\pi}{6} = \cos^{-1}\left(\cos(2\pi+\frac{\pi}{6})\right) = \cos^{-1}\left(\cos\frac{\pi}{6}\right)$

$$= \frac{\pi}{6}.$$

Example 4 Find the value of $\tan^{-1} \tan\frac{9\pi}{8}$.

Solution $\tan^{-1} \tan\frac{9\pi}{8} = \tan^{-1} \tan\left(\pi + \frac{\pi}{8}\right)$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{8}\right)\right) = \frac{\pi}{8}$$

Example 5 Evaluate $\tan(\tan^{-1}(-4))$.

Solution Since $\tan(\tan^{-1}x) = x, \forall\ x \in R$, $\tan(\tan^{-1}(-4) = -4$.

Example 6 Evaluate: $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$.

Solution $\tan^{-1}\sqrt{3} - \sec^{-1}(-2) = \tan^{-1}\sqrt{3} - [\pi - \sec^{-1}2]$

$$= \frac{\pi}{3} - \pi + \cos^{-1}\left(\frac{1}{2}\right) = -\frac{2\pi}{3} + \frac{\pi}{3} = -\frac{\pi}{3}.$$

Example 7 Evaluate: $\sin^{-1} \quad \cos \quad \sin^{-1}\frac{\sqrt{3}}{2}$.

Solution $\sin^{-1} \quad \cos \quad \sin^{-1}\frac{\sqrt{3}}{2} \qquad \sin^{-1} \quad \cos \quad \frac{\pi}{3} \quad = \sin^{-1} \quad \frac{1}{2} \qquad \frac{\pi}{6}$.

Example 8 Prove that $\tan(\cot^{-1}x) = \cot(\tan^{-1}x)$. State with reason whether the equality is valid for all values of x.

Solution Let $\cot^{-1}x = \theta$. Then $\cot\theta = x$

or, $\tan \quad \frac{\pi}{2} - \theta \quad = x \Rightarrow \tan^{-1}x = \frac{\pi}{2} - \theta$

So $\tan(\cot^{-1}x) = \tan\theta = \cot\left(\frac{\pi}{2} - \theta\right) = \cot\left(\frac{\pi}{2} - \cot^{-1}x\right) = \cot(\tan^{-1}x)$

The equality is valid for all values of x since $\tan^{-1}x$ and $\cot^{-1}x$ are true for $x \in \mathbf{R}$.

Example 9 Find the value of $\sec\left(\tan^{-1}\frac{y}{2}\right)$.

Solution Let $\tan^{-1}\frac{y}{2} = \theta$, where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So, $\tan\theta = \frac{y}{2}$,

which gives $\sec\theta = \frac{\sqrt{4 \quad y^2}}{2}$.

Therefore, $\sec\left(\tan^{-1}\frac{y}{2}\right) = \sec\theta = \frac{\sqrt{4+y^2}}{2}$.

Example 10 Find value of $\tan(\cos^{-1}x)$ and hence evaluate $\tan \quad \cos^{-1}\frac{8}{17}$.

Solution Let $\cos^{-1}x = \theta$, then $\cos\theta = x$, where $\theta \in [0, \pi]$

Therefore, $\tan(\cos^{-1}x) = \tan\theta = \dfrac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \dfrac{\sqrt{1-x^2}}{x}$.

Hence $\tan\left(\cos^{-1}\dfrac{8}{17}\right) = \dfrac{\sqrt{1-\left(\dfrac{8}{17}\right)^2}}{\dfrac{8}{17}} = \dfrac{15}{8}$.

Example 11 Find the value of $\sin\ 2\cot^{-1}\dfrac{-5}{12}$

Solution Let $\cot^{-1}\left(\dfrac{-5}{12}\right) = y$. Then $\cot y = \dfrac{-5}{12}$.

Now $\sin\ 2\cot^{-1}\dfrac{-5}{12} = \sin 2y$

$$= 2\sin y\cos y = 2\ \frac{12}{13}\ \frac{-5}{13} \qquad \left[\text{since } \cot y < 0, \text{ so } y \in \left(\frac{\pi}{2}, \pi\right)\right]$$

$$\frac{-120}{169}$$

Example 12 Evaluate $\cos\ \sin^{-1}\dfrac{1}{4}\ \ \sec^{-1}\dfrac{4}{3}$

Solution $\cos\ \sin^{-1}\dfrac{1}{4}\ \ \sec^{-1}\dfrac{4}{3} = \cos\left[\sin^{-1}\dfrac{1}{4} + \cos^{-1}\dfrac{3}{4}\right]$

$$= \cos\ \sin^{-1}\frac{1}{4}\ \cos\ \cos^{-1}\frac{3}{4} - \sin\ \sin^{-1}\frac{1}{4}\ \sin\ \cos^{-1}\frac{3}{4}$$

$$= \frac{3}{4}\sqrt{1-\ \frac{1}{4}\ ^2} - \frac{1}{4}\sqrt{1-\ \frac{3}{4}\ ^2}$$

$$= \frac{3}{4}\frac{\sqrt{15}}{4} - \frac{1}{4}\frac{\sqrt{7}}{4}\quad \frac{3\sqrt{15}-\sqrt{7}}{16}.$$

Long Answer (L.A.)

Example 13 Prove that $2\sin^{-1}\frac{3}{5} - \tan^{-1}\frac{17}{31} = \frac{\pi}{4}$

Solution Let $\sin^{-1}\frac{3}{5} = \theta$, then $\sin\theta = \frac{3}{5}$, where $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Thus $\tan\theta = \frac{3}{4}$, which gives $\theta = \tan^{-1}\frac{3}{4}$.

Therefore, $$2\sin^{-1}\frac{3}{5} - \tan^{-1}\frac{17}{31}$$

$$= 2\theta - \tan^{-1}\frac{17}{31} \quad = 2\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{17}{31}$$

$$= \tan^{-1}\left(\frac{2.\frac{3}{4}}{1-\frac{9}{16}}\right) - \tan^{-1}\frac{17}{31} = \tan^{-1}\frac{24}{7} - \tan^{-1}\frac{17}{31}$$

$$= \tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}.\frac{17}{31}}\right) = \frac{\pi}{4}$$

Example 14 Prove that

$$\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18 = \cot^{-1}3$$

Solution We have

$$\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18$$

$$= \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18} \qquad \left(\text{since } \cot^{-1}x = \tan^{-1}\frac{1}{x}, \text{ if } x > 0\right)$$

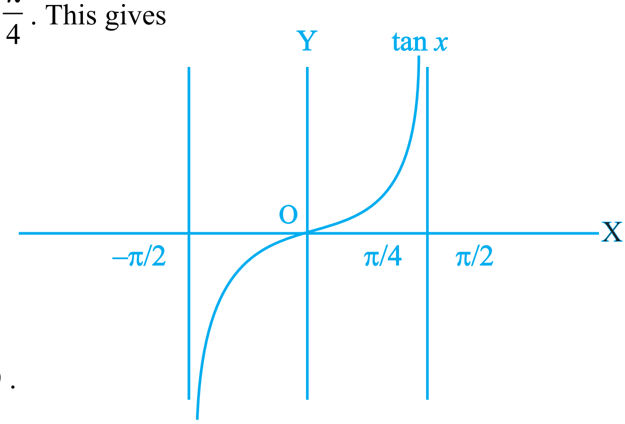
$$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}\right) + \tan^{-1}\frac{1}{18} \qquad \left(\text{since } x \,.\, y = \frac{1}{7}.\frac{1}{8} < 1\right)$$

$$= \tan^{-1}\frac{3}{11} + \tan^{-1}\frac{1}{18} = \tan^{-1}\left(\frac{\frac{3}{11}+\frac{1}{18}}{1-\frac{3}{11}\times\frac{1}{18}}\right) \qquad \text{(since } xy < 1\text{)}$$

$$= \tan^{-1}\frac{65}{195} \quad = \tan^{-1}\frac{1}{3} \quad = \cot^{-1} 3$$

Example 15 Which is greater, tan 1 or $\tan^{-1} 1$?

Solution From Fig. 2.1, we note that tan x is an increasing function in the interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, since $1 > \frac{\pi}{4} \Rightarrow \tan 1 > \tan \frac{\pi}{4}$. This gives



$$\tan 1 > 1$$

$$\Rightarrow \qquad \tan 1 > 1 > \frac{\pi}{4}$$

$$\Rightarrow \qquad \tan 1 > 1 > \tan^{-1}(1).$$

Example 16 Find the value of

$$\sin\left(2\tan^{-1}\frac{2}{3}\right) + \cos(\tan^{-1}\sqrt{3}).$$

Solution Let $\tan^{-1}\frac{2}{3} = x$ and $\tan^{-1}\sqrt{3} = y$ so that $\tan x = \frac{2}{3}$ and $\tan y = \sqrt{3}$.

Therefore, $$\sin\left(2\tan^{-1}\frac{2}{3}\right) + \cos(\tan^{-1}\sqrt{3})$$

$$= \sin(2x) + \cos y$$

$$= \frac{2\tan x}{1+\tan^2 x} + \frac{1}{\sqrt{1+\tan^2 y}} = \frac{2.\frac{2}{3}}{1+\frac{4}{9}} + \frac{1}{1+\sqrt{\left(\sqrt{3}\right)^2}}$$

$$= \frac{12}{13} + \frac{1}{2} = \frac{37}{26}.$$

Example 17 Solve for x

$$\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x,\ x>0$$

Solution From given equation, we have $2\tan^{-1}\left(\frac{1-x}{1+x}\right)=\tan^{-1}x$

$$\Rightarrow \qquad 2\left[\tan^{-1}1-\tan^{-1}x\right]=\tan^{-1}x$$

$$\Rightarrow \qquad 2\left(\frac{\pi}{4}\right)=3\tan^{-1}x \Rightarrow \frac{\pi}{6}=\tan^{-1}x$$

$$\Rightarrow \qquad x=\frac{1}{\sqrt{3}}$$

Example 18 Find the values of x which satisfy the equation

$$\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x.$$

Solution From the given equation, we have

$$\sin(\sin^{-1}x+\sin^{-1}(1-x))=\sin(\cos^{-1}x)$$

$$\Rightarrow \sin(\sin^{-1}x)\cos(\sin^{-1}(1-x))+\cos(\sin^{-1}x)\sin(\sin^{-1}(1-x))=\sin(\cos^{-1}x)$$

$$\Rightarrow x\sqrt{1-(1-x)^2}+(1-x)\sqrt{1-x^2}=\sqrt{1-x^2}$$

$$\Rightarrow x\sqrt{2x-x^2}+\sqrt{1-x^2}\,(1-x-1)=0$$

$$\Rightarrow x\left(\sqrt{2x-x^2}-\sqrt{1-x^2}\right)=0$$

$$\Rightarrow x=0 \quad \text{or} \quad 2x-x^2=1-x^2$$

$$\Rightarrow x=0 \quad \text{or} \quad x=\frac{1}{2}.$$

Example 19 Solve the equation $\sin^{-1}6x+\sin^{-1}6\sqrt{3}\,x=-\frac{\pi}{2}$

Solution From the given equation, we have $\sin^{-1}6x=-\frac{\pi}{2}-\sin^{-1}6\sqrt{3}\,x$

$$\Rightarrow \quad \sin(\sin^{-1} 6x) = \sin\left(-\frac{\pi}{2} - \sin^{-1} 6\sqrt{3}\,x\right)$$

$$\Rightarrow \quad 6x = -\cos(\sin^{-1} 6\sqrt{3}\,x)$$

$$\Rightarrow \quad 6x = -\sqrt{1-108x^2}$$. Squaring, we get

$$36x^2 = 1 - 108x^2$$

$$\Rightarrow \quad 144x^2 = 1 \qquad \Rightarrow x = \pm\frac{1}{12}$$

Note that $x = -\frac{1}{12}$ is the only root of the equation as $x = \frac{1}{12}$ does not satisfy it.

Example 20 Show that

$$2\tan^{-1}\left\{\tan\frac{\alpha}{2}.\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right\} = \tan^{-1}\frac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}$$

Solution L.H.S. $= \tan^{-1}\dfrac{2\tan\frac{\alpha}{2}.\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}{1-\tan^2\frac{\alpha}{2}\tan^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}$ $\left(\text{since } 2\tan^{-1} x = \tan^{-1}\dfrac{2x}{1-x^2}\right)$

$$= \tan^{-1}\frac{2\tan\frac{\alpha}{2}\,\dfrac{1-\tan\frac{\beta}{2}}{1+\tan\frac{\beta}{2}}}{1-\tan^2\frac{\alpha}{2}\left(\dfrac{1-\tan\frac{\beta}{2}}{1+\tan\frac{\beta}{2}}\right)^2}$$

$$= \tan^{-1}\frac{2\tan\frac{\alpha}{2}.\left(1-\tan^2\frac{\beta}{2}\right)}{\left(1+\tan\frac{\beta}{2}\right)^2 - \tan^2\frac{\alpha}{2}\left(1-\tan\frac{\beta}{2}\right)^2}$$

$$= \tan^{-1}\frac{2\tan\frac{\alpha}{2}\left(1-\tan^2\frac{\beta}{2}\right)}{\left(1+\tan^2\frac{\beta}{2}\right)\left(1-\tan^2\frac{\alpha}{2}\right)+2\tan\frac{\beta}{2}\left(1+\tan^2\frac{\alpha}{2}\right)}$$

$$= \tan^{-1}\frac{\dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}\;\dfrac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}{\dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}+\dfrac{2\tan\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}$$

$$= \tan^{-1}\left(\frac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}\right) = \text{R.H.S.}$$

Objective type questions

Choose the correct answer from the given four options in each of the Examples 21 to 41.

Example 21 Which of the following corresponds to the principal value branch of $\tan^{-1}$?

(A) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (B) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

(C) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right) - \{0\}$ (D) $(0, \pi)$

Solution (A) is the correct answer.

Example 22 The principal value branch of $\sec^{-1}$ is

(A) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$ (B) $[0, \pi]-\left\{\frac{\pi}{2}\right\}$

(C) $(0, \pi)$ (D) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Solution (B) is the correct answer.

Example 23 One branch of $\cos^{-1}$ other than the principal value branch corresponds to

(A) $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ (B) $[\pi, 2\pi] - \left\{\frac{3\pi}{2}\right\}$

(C) $(0, \pi)$ (D) $[2\pi, 3\pi]$

Solution (D) is the correct answer.

Example 24 The value of $\sin^{-1}\left(\cos\left(\frac{43\pi}{5}\right)\right)$ is

(A) $\frac{3\pi}{5}$ (B) $\frac{-7\pi}{5}$ (C) $\frac{\pi}{10}$ (D) $-\frac{\pi}{10}$

Solution (D) is the correct answer. $\sin^{-1}\left(\cos\frac{40\pi + 3\pi}{5}\right) = \sin^{-1}\cos\left(8\pi + \frac{3\pi}{5}\right)$

$$= \sin^{-1}\left(\cos\frac{3\pi}{5}\right) = \sin^{-1}\left(\sin\left(\frac{\pi}{2} - \frac{3\pi}{5}\right)\right)$$

$$= \sin^{-1}\left(\sin\left(-\frac{\pi}{10}\right)\right) = -\frac{\pi}{10}.$$

Example 25 The principal value of the expression $\cos^{-1}[\cos(-680°)]$ is

(A) $\frac{2\pi}{9}$ (B) $\frac{-2\pi}{9}$ (C) $\frac{34\pi}{9}$ (D) $\frac{\pi}{9}$

Solution (A) is the correct answer. $\cos^{-1}(\cos(680°)) = \cos^{-1}[\cos(720° - 40°)]$

$$= \cos^{-1}[\cos(-40°)] = \cos^{-1}[\cos(40°)] = 40° = \frac{2\pi}{9}.$$

Example 26 The value of $\cot(\sin^{-1}x)$ is

(A) $\frac{\sqrt{1+x^2}}{x}$ (B) $\frac{x}{\sqrt{1+x^2}}$

(C) $\frac{1}{x}$ (D) $\frac{\sqrt{1-x^2}}{x}$.

Solution (D) is the correct answer. Let $\sin^{-1} x = \theta$, then $\sin\theta = x$

$$\Rightarrow \quad \text{cosec } \theta = \frac{1}{x} \Rightarrow \text{cosec}^2\theta = \frac{1}{x^2}$$

$$\Rightarrow \quad 1 + \cot^2 \theta = \frac{1}{x^2} \Rightarrow \cot\theta = \frac{\sqrt{1-x^2}}{x}.$$

Example 27 If $\tan^{-1}x = \frac{\pi}{10}$ for some $x \in \mathbf{R}$, then the value of $\cot^{-1}x$ is

(A) $\frac{\pi}{5}$ (B) $\frac{2\pi}{5}$ (C) $\frac{3\pi}{5}$ (D) $\frac{4\pi}{5}$

Solution (B) is the correct answer. We know $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$. Therefore

$$\cot^{-1}x = \frac{\pi}{2} - \frac{\pi}{10}$$

$$\Rightarrow \cot^{-1}x = \frac{\pi}{2} - \frac{\pi}{10} = \frac{2\pi}{5}.$$

Example 28 The domain of $\sin^{-1} 2x$ is

(A) [0, 1] (B) [– 1, 1]

(C) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (D) [–2, 2]

Solution (C) is the correct answer. Let $\sin^{-1}2x = \theta$ so that $2x = \sin \theta$.

Now $-1 \le \sin \theta \le 1$, i.e., $-1 \le 2x \le 1$ which gives $-\frac{1}{2} \le x \le \frac{1}{2}$.

Example 29 The principal value of $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ is

(A) $-\frac{2\pi}{3}$ (B) $-\frac{\pi}{3}$ (C) $\frac{4\pi}{3}$ (D) $\frac{5\pi}{3}$.

Solution (B) is the correct answer.

$$\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)=\sin^{-1}\left(-\sin\frac{\pi}{3}\right)=-\sin^{-1}\left(\sin\frac{\pi}{3}\right)=-\frac{\pi}{3}.$$

Example 30 The greatest and least values of $(\sin^{-1}x)^2 + (\cos^{-1}x)^2$ are respectively

(A) $\frac{5\pi^2}{4}$ and $\frac{\pi^2}{8}$ (B) $\frac{\pi}{2}$ and $\frac{-\pi}{2}$

(C) $\frac{\pi^2}{4}$ and $\frac{-\pi^2}{4}$ (D) $\frac{\pi^2}{4}$ and 0.

Solution (A) is the correct answer. We have

$$(\sin^{-1}x)^2 + (\cos^{-1}x)^2 = (\sin^{-1}x + \cos^{-1}x)^2 - 2\sin^{-1}x\,\cos^{-1}x$$

$$= \frac{\pi^2}{4} - 2\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right)$$

$$= \frac{\pi^2}{4} - \pi\sin^{-1}x + 2\left(\sin^{-1}x\right)^2$$

$$= 2\left[\left(\sin^{-1}x\right)^2 - \frac{\pi}{2}\sin^{-1}x + \frac{\pi^2}{8}\right]$$

$$= 2\left[\left(\sin^{-1}x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16}\right].$$

Thus, the least value is $2\left(\frac{\pi^2}{16}\right)$ i.e. $\frac{\pi^2}{8}$ and the Greatest value is $2\left[\left(\frac{-\pi}{2} - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16}\right]$,

i.e. $\frac{5\pi^2}{4}$.

Example 31 Let $\theta = \sin^{-1}(\sin(-600°))$, then value of θ is

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\frac{2\pi}{3}$ (D) $\frac{-2\pi}{3}$.

Solution (A) is the correct answer.

$$\sin^{-1}\sin\left(-600\times\frac{\pi}{180}\right)=\sin^{-1}\sin\left(\frac{-10\pi}{3}\right)$$

$$=\sin^{-1}\left[-\sin\left(4\pi-\frac{2\pi}{3}\right)\right]=\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$$

$$=\sin^{-1}\left(\sin\left(\pi-\frac{\pi}{3}\right)\right)=\sin^{-1}\left(\sin\frac{\pi}{3}\right)=\frac{\pi}{3}.$$

Example 32 The domain of the function $y = \sin^{-1}(-x^2)$ is

(A) [0, 1] (B) (0, 1)
(C) [–1, 1] (D) ϕ

Solution (C) is the correct answer. $y = \sin^{-1}(-x^2) \Rightarrow \sin y = -x^2$

i.e. $-1 \le -x^2 \le 1$ (since $-1 \le \sin y \le 1$)

$\Rightarrow 1 \ge x^2 \ge -1$

$\Rightarrow 0 \le x^2 \le 1$

$\Rightarrow |x| \le 1$ *i.e.* $-1 \le x \le 1$

Example 33 The domain of $y = \cos^{-1}(x^2 - 4)$ is

(A) [3, 5] (B) [0, π]

(C) $\left[-\sqrt{5},-\sqrt{3}\right]\cap\left[-\sqrt{5},\sqrt{3}\right]$ (D) $\left[-\sqrt{5},-\sqrt{3}\right]\cup\left[\sqrt{3},\sqrt{5}\right]$

Solution (D) is the correct answer. $y = \cos^{-1}(x^2 - 4) \Rightarrow \cos y = x^2 - 4$

i.e. $-1 \le x^2 - 4 \le 1$ (since $-1 \le \cos y \le 1$)

$\Rightarrow 3 \le x^2 \le 5$

$\Rightarrow \sqrt{3} \le |x| \le \sqrt{5}$

$\Rightarrow x \in \left[-\sqrt{5},-\sqrt{3}\right]\cup\left[\sqrt{3},\sqrt{5}\right]$

Example 34 The domain of the function defined by $f(x) = \sin^{-1}x + \cos x$ is

(A) [–1, 1] (B) [–1, π + 1]

(C) $(-\infty, \infty)$ (D) ϕ

Solution (A) is the correct answer. The domain of cos is **R** and the domain of $\sin^{-1}$ is [–1, 1]. Therefore, the domain of $\cos x + \sin^{-1}x$ is $\mathbf{R} \cap [-1,1]$, i.e., [–1, 1].

Example 35 The value of sin (2 $\sin^{-1}$ (·6)) is

(A) ·48 (B) ·96 (C) 1·2 (D) sin 1·2

Solution (B) is the correct answer. Let $\sin^{-1}$ (·6) = θ, i.e., sin θ = ·6.

Now sin (2θ) = 2 sinθ cosθ = 2 (·6) (·8) = ·96.

Example 36 If $\sin^{-1} x + \sin^{-1} y = \frac{\pi}{2}$, then value of $\cos^{-1} x + \cos^{-1} y$ is

(A) $\frac{\pi}{2}$ (B) π (C) 0 (D) $\frac{2\pi}{3}$

Solution (A) is the correct answer. Given that $\sin^{-1} x + \sin^{-1} y = \frac{\pi}{2}$.

Therefore, $\left(\frac{\pi}{2} - \cos^{-1} x\right) + \left(\frac{\pi}{2} - \cos^{-1} y\right) = \frac{\pi}{2}$

$$\Rightarrow \qquad \cos^{-1}x + \cos^{-1}y = \frac{\pi}{2}.$$

Example 37 The value of $\tan\left(\cos^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4}\right)$ is

(A) $\frac{19}{8}$ (B) $\frac{8}{19}$ (C) $\frac{19}{12}$ (D) $\frac{3}{4}$

Solution (A) is the correct answer. $\tan\left(\cos^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4}\right) = \tan\left(\tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{4}\right)$

$$= \tan \tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{4}}{1-\frac{4}{3}\times\frac{1}{4}}\right) = \tan \tan^{-1}\left(\frac{19}{8}\right) = \frac{19}{8}.$$

Example 38 The value of the expression sin [cot^{-1} (cos (tan^{-1} 1))] is

(A) 0 (B) 1 (C) $\frac{1}{\sqrt{3}}$ (D) $\sqrt{\frac{2}{3}}$.

Solution (D) is the correct answer.

$$\sin [\cot^{-1} (\cos \frac{\pi}{4})] = \sin [\cot^{-1} \frac{1}{\sqrt{2}}] = \sin\left[\sin^{-1}\sqrt{\frac{2}{3}}\right] = \sqrt{\frac{2}{3}}$$

Example 39 The equation $\tan^{-1}x - \cot^{-1}x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ has

(A) no solution (B) unique solution
(C) infinite number of solutions (D) two solutions

Solution (B) is the correct answer. We have

$$\tan^{-1}x - \cot^{-1}x = \frac{\pi}{6} \text{ and } \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$$

Adding them, we get $2\tan^{-1}x = \frac{2\pi}{3}$

$$\Rightarrow \tan^{-1}x = \frac{\pi}{3} \text{ i.e., } x = \sqrt{3}.$$

Example 40 If $\alpha \le 2 \sin^{-1}x + \cos^{-1}x \le \beta$, then

(A) $\alpha = \frac{-\pi}{2}, \beta = \frac{\pi}{2}$ (B) $\alpha = 0, \beta = \pi$

(C) $\alpha = \frac{-\pi}{2}, \beta = \frac{3\pi}{2}$ (D) $\alpha = 0, \beta = 2\pi$

Solution (B) is the correct answer. We have $\frac{-\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$

$$\Rightarrow \qquad \frac{-\pi}{2} + \frac{\pi}{2} \leq \sin^{-1}x + \frac{\pi}{2} \leq \frac{\pi}{2} + \frac{\pi}{2}$$

$$\Rightarrow \qquad 0 \leq \sin^{-1}x + (\sin^{-1}x + \cos^{-1}x) \leq \pi$$

$$\Rightarrow \qquad 0 \leq 2\sin^{-1}x + \cos^{-1}x \leq \pi$$

Example 41 The value of $\tan^2 (\sec^{-1}2) + \cot^2 (\text{cosec}^{-1}3)$ is

(A) 5 (B) 11 (C) 13 (D) 15

Solution (B) is the correct answer.

$\tan^2 (\sec^{-1}2) + \cot^2 (\text{cosec}^{-1}3) = \sec^2 (\sec^{-1}2) - 1 + \text{cosec}^2 (\text{cosec}^{-1}3) - 1$

$= 2^2 \times 1 + 3^2 - 2 = 11.$

2.3 EXERCISE

Short Answer (S.A.)

1. Find the value of $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

2. Evaluate $\cos\ \ \cos^{-1}\ \frac{-\sqrt{3}}{2}\ \ \ \frac{}{6}$.

3. Prove that $\cot\ \ \frac{}{4}-2\cot^{-1}3\ \ \ 7$.

4. Find the value of $\tan^{-1}\ -\frac{1}{\sqrt{3}}\ \ \ \cot^{-1}\frac{1}{\sqrt{3}}\ \ \ \tan^{-1}\ \sin\ \frac{-}{2}$.

5. Find the value of $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$.

6. Show that $2\tan^{-1}(-3) = \frac{-}{2} + \tan^{-1}\left(\frac{-4}{3}\right)$.

7. Find the real solutions of the equation

$$\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}.$$

8. Find the value of the expression $\sin\left(2\tan^{-1}\frac{1}{3}\right)+\cos\left(\tan^{-1}2\sqrt{2}\right)$.

9. If 2 $\tan^{-1}$ ($\cos\theta$) = $\tan^{-1}$ (2 cosec θ), then show that $\theta = \frac{\pi}{4}$,

where n is any integer.

10. Show that $\cos\left(2\tan^{-1}\frac{1}{7}\right)=\sin\left(4\tan^{-1}\frac{1}{3}\right)$.

11. Solve the following equation $\cos\left(\tan^{-1}x\right)=\sin\left(\cot^{-1}\frac{3}{4}\right)$.

Long Answer (L.A.)

12. Prove that $\tan^{-1}\frac{\sqrt{1 \quad x^2} \quad \sqrt{1-x^2}}{\sqrt{1 \quad x^2}-\sqrt{1-x^2}} \quad \overline{4} \quad \frac{1}{2}\cos^{-1}x^2$

13. Find the simplified form of $\cos^{-1}\ \frac{3}{5}\cos x \quad \frac{4}{5}\sin x$, where $x \in \ \frac{-3}{4}, \overline{4}$.

14. Prove that $\sin^{-1}\frac{8}{17} \quad \sin^{-1}\frac{3}{5} \quad \sin^{-1}\frac{77}{85}$.

15. Show that $\sin^{-1}\frac{5}{13} \quad \cos^{-1}\frac{3}{5} \quad \tan^{-1}\frac{63}{16}$.

16. Prove that $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\sin^{-1}\frac{1}{\sqrt{5}}$.

17. Find the value of $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}$.

18. Show that $\tan \frac{1}{2}\sin^{-1}\frac{3}{4} \quad \frac{4-\sqrt{7}}{3}$ and justify why the other value $\frac{4+\sqrt{7}}{3}$ is ignored?

19. If $a_1, a_2, a_3, \ldots, a_n$ is an arithmetic progression with common difference d, then evaluate the following expression.

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1 a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2 a_3}\right)+\tan^{-1}\left(\frac{d}{1+a_3 a_4}\right)+\ldots+\tan^{-1}\left(\frac{d}{1+a_{n-1} a_n}\right)\right].$$

Objective Type Questions

Choose the correct answers from the given four options in each of the Exercises from 20 to 37 (M.C.Q.).

20. Which of the following is the principal value branch of $\cos^{-1}x$?

(A) $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ (B) $(0, \pi)$

(C) $[0, \pi]$ (D) $(0, \pi) - \left\{\frac{\pi}{2}\right\}$

21. Which of the following is the principal value branch of $\text{cosec}^{-1}x$?

(A) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ (B) $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

(C) $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ (D) $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

22. If $3\tan^{-1} x + \cot^{-1} x = \pi$, then x equals

(A) 0 (B) 1 (C) –1 (D) $\frac{1}{2}$.

23. The value of $\sin^{-1} \quad \cos \frac{33}{5}$ is

(A) $\frac{3\pi}{5}$ (B) $\frac{-7\pi}{5}$ (C) $\frac{\pi}{10}$ (D) $\frac{-\pi}{10}$

24. The domain of the function $\cos^{-1}(2x - 1)$ is

(A) $[0, 1]$ (B) $[-1, 1]$

(C) $(-1, 1)$ (D) $[0, \pi]$

25. The domain of the function defined by $f(x) = \sin^{-1}\sqrt{x-1}$ is

(A) $[1, 2]$ (B) $[-1, 1]$

(C) $[0, 1]$ (D) none of these

26. If $\cos\left(\sin^{-1}\frac{2}{5} + \cos^{-1} x\right) = 0$, then x is equal to

(A) $\frac{1}{5}$ (B) $\frac{2}{5}$ (C) 0 (D) 1

27. The value of $\sin(2\tan^{-1}(.75))$ is equal to

(A) ·75 (B) 1·5 (C) ·96 (D) sin 1·5

28. The value of $\cos^{-1} \quad \cos\frac{3}{2}$ is equal to

(A) $\frac{\pi}{2}$ (B) $\frac{3\pi}{2}$ (C) $\frac{5\pi}{2}$ (D) $\frac{7\pi}{2}$

29. The value of the expression $2\sec^{-1} 2 + \sin^{-1} \quad \frac{1}{2}$ is

(A) $\frac{\pi}{6}$ (B) $\frac{5\pi}{6}$ (C) $\frac{7\pi}{6}$ (D) 1

30. If $\tan^{-1} x + \tan^{-1} y = \frac{4\pi}{5}$, then $\cot^{-1} x + \cot^{-1} y$ equals

(A) $\frac{\pi}{5}$ (B) $\frac{2\pi}{5}$ (C) $\frac{3}{5}$ (D) π

31. If $\sin^{-1} \frac{2a}{1 \quad a^2} \quad \cos^{-1} \frac{1-a^2}{1 \quad a^2} \quad \tan^{-1} \frac{2x}{1-x^2}$, where $a, x \in]0, 1$, then the value of x is

(A) 0 (B) $\frac{a}{2}$ (C) a (D) $\frac{2a}{1-a^2}$

32. The value of cot $\cos^{-1}\frac{7}{25}$ is

(A) $\frac{25}{24}$ (B) $\frac{25}{7}$ (C) $\frac{24}{25}$ (D) $\frac{7}{24}$

33. The value of the expression tan $\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}$ is

(A) $2 \quad \sqrt{5}$ (B) $\sqrt{5}-2$

(C) $\frac{\sqrt{5} \quad 2}{2}$ (D) $5 \quad \sqrt{2}$

$$\left[\text{Hint}: \tan\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right]$$

34. If $|x| \leq 1$, then $2\tan^{-1}x + \sin^{-1}\frac{2x}{1 \quad x^2}$ is equal to

(A) $4\tan^{-1}x$ (B) 0 (C) $\frac{\ }{2}$ (D) π

35. If $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$, then $\alpha(\beta+\gamma) + \beta(\gamma+\alpha) + \gamma(\alpha+\beta)$ equals

(A) 0 (B) 1 (C) 6 (D) 12

36. The number of real solutions of the equation $\sqrt{1+\cos 2x}=\sqrt{2}\cos^{-1}(\cos x)$ in $\left[\frac{\pi}{2},\pi\right]$ is

(A) 0 (B) 1 (C) 2 (D) Infinite

37. If $\cos^{-1}x > \sin^{-1}x$, then

(A) $\frac{1}{\sqrt{2}}<x\leq 1$ (B) $0 \leq x < \frac{1}{\sqrt{2}}$

(C) $-1\leq x<\frac{1}{\sqrt{2}}$ (D) $x > 0$

Fill in the blanks in each of the Exercises 38 to 48.

38. The principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is__________.

39. The value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$ is__________.

40. If $\cos(\tan^{-1} x + \cot^{-1}\sqrt{3}) = 0$, then value of x is__________.

41. The set of values of $\sec^{-1}\left(\frac{1}{2}\right)$ is__________.

42. The principal value of $\tan^{-1}\sqrt{3}$ is__________.

43. The value of $\cos^{-1}\left(\cos\frac{14\pi}{3}\right)$ is__________.

44. The value of $\cos(\sin^{-1} x + \cos^{-1} x)$, $|x| \le 1$ is______ .

45. The value of expression $\tan\left(\frac{\sin^{-1} x + \cos^{-1} x}{2}\right)$, when $x = \frac{\sqrt{3}}{2}$ is__________.

46. If $y = 2\tan^{-1} x + \sin^{-1}\frac{2x}{1\quad x^2}$ for all x, then____ $< y <$____.

47. The result $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ is true when value of xy is _____

48. The value of $\cot^{-1}(-x)$ for all $x \in \mathbf{R}$ in terms of $\cot^{-1}x$ is _______.

State **True** or **False** for the statement in each of the Exercises 49 to 55.

49. All trigonometric functions have inverse over their respective domains.

50. The value of the expression $(\cos^{-1} x)^2$ is equal to $\sec^2 x$.

51. The domain of trigonometric functions can be restricted to any one of their branch (not necessarily principal value) in order to obtain their inverse functions.

52. The least numerical value, either positive or negative of angle θ is called principal value of the inverse trigonometric function.

53. The graph of inverse trigonometric function can be obtained from the graph of their corresponding trigonometric function by interchanging x and y axes.

54. The minimum value of n for which $\tan^{-1}\frac{n}{\pi}>\frac{\pi}{4}, n\in\mathbf{N}$, is valid is 5.

55. The principal value of $\sin^{-1}\left[\cos\left(\sin^{-1}\frac{1}{2}\right)\right]$ is $\frac{\pi}{3}$.